

# Imperfect best-response mechanisms\*

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## Abstract

Best-response mechanisms (Nisan, Schapira, Valiant, Zohar, 2011) provide a unifying framework for studying various distributed protocols in which the participants are instructed to repeatedly best respond to each others' strategies. Two fundamental features of these mechanisms are convergence and incentive compatibility.

This work investigates convergence and incentive compatibility conditions of such mechanisms when players are not guaranteed to always best respond but they rather play an *imperfect* best-response strategy. That is, at every time step every player deviates from the prescribed best-response strategy according to some probability parameter. The results explain to what extent convergence and incentive compatibility depend on the assumption that players never make mistakes, and how robust such protocols are to “noise” or “mistakes”.

## 1 Introduction

In many distributed protocols the participants, termed *players*, have to play (can be seen as playing) some underlying base game over and over (or until some equilibrium point is reached). Hence, an appealing theoretical model for describing these protocols is provided by *game dynamics*. Nisan *et al* [NSVZ11] introduce a class of game dynamics, called *best-response mechanisms*, in which the players are instructed to always best-respond to what the other players are currently doing. They identify an interesting class of games for which the resulting dynamics satisfies:

- *Convergence*. The dynamics eventually reaches a unique equilibrium point (a unique pure Nash equilibrium) of the base game regardless of the order in which players respond and of the presence of concurrent responses.

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- *Incentive compatibility.* A player who deviates from the prescribed best-response strategy can only worsen his/her final utility, that is, the dynamics will reach a different equilibrium that yields weakly smaller payoff.

Note that convergence and incentive compatibility say that the protocol will eventually “stabilize” if implemented correctly, and that the participants are actually willing to do so. The class for which these features has been proved is given by games for which a Nash equilibrium is computed by iteratively eliminating “useless” strategies, called *never best-response* (NBR) strategies. In fact, Nisan et al [NSVZ11] showed that this class of games captures several protocols and mechanisms arising in computerized and economics settings: (1) the Border Gateway Protocol (BGP) currently used in the Internet, (2) a game-theoretic version of the TCP protocol, and (3) mechanisms for the classical cost-sharing and stable roommates problems studied in micro economics.

In this work we address the following question:

*What happens if players do not always best respond?*

Is it possible that when players “occasionally” deviate from the prescribed protocol (e.g., by making mistakes in computing their best-response) then the protocol does not converge anymore? Can such mistakes induce some other player to adopt a “non-best-response” strategy that results in a better payoff?

**Our contribution** This work investigates convergence and incentive compatibility conditions of the best-response dynamics/mechanisms in [NSVZ11] when players are not guaranteed to always best respond but they rather play an *imperfect* best-response strategy. That is, at every time step every player deviates from the prescribed best-response strategy according to some probability parameter  $p \geq 0$ . The parameter  $p$  can be regarded as the probability of making a “mistake” every time the player updates his/her strategy. We prove the following results:

- *Convergence.* The convergence to the pure Nash equilibrium may not occur even for  $p$  being exponentially small in the number  $n$  of players (Theorem 6). Such negative result applies also to certain instances of the BGP games. This negative result is complemented by a general positive result saying that  $p$  needs to be polynomially small with respect to some parameters defining the schedule of the players (Theorem 8). This gives also a bound on the *time* needed to converge which just a bit more than the upper bound in [NSVZ11] for “perfect” best-response ( $p = 0$ ). Note that in our setting we assume a reasonably weaker adversarial schedule (Definition 4).
- *Incentive compatibility.* We show that this feature requires a slightly stronger condition than the one given in [NSVZ11] which takes into account the parameter  $p$  and, essentially, the possibility that the other players do not “completely” discard their NBR strategies.

- *Generalized games and equilibria.* We also consider a *more general* class of games in which the elimination of NBR strategies will only result in a *subgame* (and not necessarily in a unique strategy profile). In this case, when  $p$  is small enough, the dynamics is essentially the dynamics of the subgame and thus equilibrium of the subgame provides a good description of the equilibrium of the dynamics, regardless of the kind of equilibrium at which one is interested. Furthermore, the time to reach such an equilibrium is the sum the time needed to “converge to the subgame” plus the time needed to reach the equilibrium by the dynamics that runs on the subgame only (Theorem 15).

These results indicate to what extent convergence and incentive compatibility depend on the assumption that players never make mistakes, and how robust such protocols are to “noise” or “mistakes”.

**Further related work.** Our imperfect best response dynamics are essentially equivalent to the *mutation model* by Kandori et al [KMR93], and to the *mistakes model* by Young [You93], and Kandori and Rob [KR95]. A related model is the *logit response* dynamics of Blume [Blu93] in which the probability of a mistake depends on payoffs of the game. The dynamics studied in these works are based on a specific schedule of the players (the order in which they play in the dynamics). Whether such an assumption effects the selected equilibrium is the main focus of a recent work by Alós-Ferrer and Netzer [AFN10].

## 2 Definitions

**Games.** We consider an  $n$ -player game in which each player  $i$  has finite strategy set  $S_i$  and utility function  $u_i$ . Sometimes we assume that each player has also a *tie breaking rule*  $\prec_i$ , i.e., a total order on  $S_i$ , that depends solely on the player’s private information: such tie-breaking rule can be implemented in a game by means of opportune perturbations of the utility function. Let us now recall some definitions from [NSVZ11].

**Definition 1** (Never Best Response). *A strategy  $s_i$  is a never best-response (NBR) for player  $i$  if, for every  $s_{-i}$ , there exists  $s'_i$  such that  $u_i(s_i, s_{-i}) < u_i(s'_i, s_{-i})$ . In the case that a tie breaking rule  $\prec_i$  has been defined for player  $i$ , then  $s_i$  is a NBR for  $i$  also if  $u_i(s'_i, s_{-i}) = u_i(s_i, s_{-i})$  and  $s_i \prec_i s'_i$ .*

**Definition 2** (Elimination Sequence). *An elimination sequence for a game  $G$  consists of a sequence of subgames*

$$G = G_0 \supset G_1 \supset \cdots \supset G_r = \hat{G},$$

*where any game  $G_{k+1}$  is obtained from the previous one by letting some player  $i^{(k)}$  eliminating strategies which are NBR in  $G_k$ .*

The length of the shortest elimination sequence for a game  $G$  is denoted with  $e_G$  (we omit the subscript when it is clear from the context). It is easy to see that for each game  $e_G \leq n(m-1)$ , where  $m$  is the maximum number of strategies of a player. Our results will focus on the following classes of games.

**Definition 3** (NBR-Reducible and NBR-Solvable Games). *The game  $G$  is NBR-reducible to  $\hat{G}$  if there exists an elimination sequence for  $G$  that ends in  $\hat{G}$ . The game  $G$  is NBR-solvable if it is NBR-reducible to  $\hat{G}$  and  $\hat{G}$  consists of an unique profile.*

As an example, consider a 2-player game, in which each player has strategy set  $\{0, 1, 2\}$  and utilities as follows:

	0	1	2
0	0,0	0,0	0,-2
1	0,0	-1,-1	-1,-2
2	-2,0	-2,-1	-2,-2

(1)

Notice that strategy 2 is a NBR for both players. Hence, there exists an elimination sequence of length 1 that reduces above game in its “upper-left”  $2 \times 2$  subgame with strategy set  $\{0, 1\}$  for each player. Therefore, this game is NBR-reducible. If we add the tie-breaking rule “prefer strategies with smaller index”, then the game reduces further to the profile  $(0, 0)$  and hence it is NBR-solvable.

**Dynamics.** A dynamics is usually specified by two rules: a *selection rule*, that specifies for each time step the subset of players that are selected for updating their strategies; an *update rule*, that specifies how a player updates her strategy (possibly depending on the past history and on the current strategy profile). In this work we focus on the following classes of selection and update rules.

**Definition 4** ( $(R, \varepsilon)$ -Fair Selection Rule). *A selection rule is  $(R, \varepsilon)$ -fair if there exists a nonnegative integer  $R$  such that, for any interval of  $R$  time steps, all players are selected at least once in this interval with probability at least  $1 - \varepsilon$ .*

As an example, scheduling players in round-robin fashion or concurrently are obviously  $(R, 0)$ -fair selection rules with  $R = n$  and  $R = 1$ , respectively, whereas selecting a player at random at each time step is  $(R, \varepsilon)$ -fair with  $R = \mathcal{O}(n \log n)$ . Observe that if a selection rule is  $(R, \varepsilon)$ -fair, then, for every  $\delta > 0$ , all players are selected at least once with probability at least  $1 - \delta$  in an interval of  $R \cdot \left\lceil \frac{\log(1/\delta)}{\log(1/\varepsilon)} \right\rceil$  time steps (this holds because the probability  $1 - \varepsilon$  is guaranteed for any interval of  $R$  time steps). We also denote with  $\eta$  the maximum number of players selected for update in one step by the selection rule. Note that  $\eta \leq n$ .

As for the update rule, we give the following definition.

**Definition 5** ( $p$ -Imperfect Update Rule). *In a  $p$ -imperfect update rule each player updates her strategy to a NBR with probability at most  $p$ .*

As an example, best-response update rule is 0-imperfect, whereas the logit update rule [Blu93] (see Appendix A for a brief overview) is  $p$ -imperfect with

$$p \leq \frac{m-1}{m-1+e^\beta}$$

for all games in which the payoff between a non-best and a best-response differ by at least one<sup>1</sup> and each player has at most  $m$  strategies.

Henceforth, we always refer as *imperfect best-response dynamics* to any dynamics whose selection rule is  $(R, \varepsilon)$ -fair and whose update rule is  $p$ -imperfect. We highlight that we do not put any other constraint on the way the dynamics run. In particular we allow both the selection rule and the update rule to depend on the *status* of the game, that is on a set of informations other than the current strategy profile.

### 3 NBR-solvable games

#### 3.1 A negative result

In this section we will show that the result about convergence of the best-response dynamics in NBR-solvable games given in [NSVZ11] is not resistant to the introduction of “noise”, i.e., there is a NBR-solvable games and an imperfect best-response dynamics that never converges to the Nash equilibrium even for values of  $p$  very small. Specifically we will prove the following theorem.

**Theorem 6.** *There exists a  $n$ -player NBR-solvable game  $G$  and an imperfect best-response dynamics with parameter  $p$  exponentially small in  $n$  such that, for every integer  $t > 0$  and for every  $0 < \varepsilon < 1$ , the dynamics is in the Nash equilibrium of  $G$  after  $t$  steps with probability at most  $\varepsilon$ .*

**The game.** Consider a NBR-solvable game with  $n$  players and two strategies 0 and 1. The elimination sequence consists of players  $1, 2, \dots, n$  eliminating strategy 0 one-by-one in this order (note that 1 is a dominant strategy for player 1 and, more in general, strategy 1 is dominant for  $i$  in the subgame in which all players  $1, \dots, i-1$  have eliminated 0). The subgame  $\hat{G}$  consists of the unique PNE that is the profile  $\mathbf{1} = (1, \dots, 1)$ .

**The  $p$ -imperfect update rule.** All players play the following  $p$ -imperfect update rule:

- Player  $i$  chooses strategy 1 with probability  $p$  if all players  $j < i$  are playing strategy 1;
- Player  $i$  chooses strategy 0 with probability  $1 - q$  if at least one player  $j < i$  is playing strategy 0, where  $0 < q \ll p$ .

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<sup>1</sup>When the minimum difference is  $\delta$  this extends easily by taking  $\beta_\delta = \beta \cdot \delta$  in place of  $\beta$ .

**The  $(2^{n-1}, 0)$ -fair selection rule.** Let us start by defining sequences  $\sigma_i$ , with  $i = 1, \dots, n$ , recursively as follows

$$\sigma_1 = 1, \quad \sigma_2 = 12, \quad \sigma_3 = 1213, \quad \dots \quad \sigma_i = \sigma_{i-1}\sigma_{i-2} \cdots \sigma_1 i.$$

Observe that each sequence has length  $2^{i-1}$ . The selection rule schedules players one at a time according to  $\sigma_n$  and then repeat.

A key observation about this selection rule is in order.

**Observation 7.** *Between any two occurrences of player  $i < n$  there is an occurrence of some player  $j > i$ .*

Intuitively speaking, this property causes any *bad move* of some player in the sequence  $\sigma_n$  to propagate to the last player  $n$ , where by “bad move” we mean that at time  $t$  the corresponding player  $\sigma_n(t)$  plays strategy 0 given that each player  $j < \sigma_n(t)$  plays 1 (thus, a bad move occurs with probability  $p$ ).

**Proof of Theorem 6.** Throughout the proof we will denote  $2^{n-1}$  as  $\tau$  for sake of readability.

Let  $X_t$  be the random variable that represents the profile of the game at step  $t$ . We will denote with  $X_t^n$  the  $n$ -th coordinate of  $X_t$ , i.e., the strategy played by player  $n$  at time  $t$ . Suppose that player  $n$  plays 0 at the beginning. Then, for every  $t < \tau$ , the probability that at time step  $t$  the game is in a Nash equilibrium is obviously 0. Consider now  $t \geq \tau$ . The probability that at time step  $t$  the game is in a Nash equilibrium is obviously less than the probability that  $X_t^n = 1$ . Hence it will be sufficient to show that  $\Pr(X_t^n = 1) \leq \varepsilon$ . Note that  $X_t^n = X_{c\tau}^n$ ,  $c$  being the largest integer such that  $t \geq c \cdot \tau$ . Since both the update rule and the selection rule described above are memoryless, for every profile  $\mathbf{x}$

$$\Pr(X_{c\tau}^n = 1 \mid X_{(c-1)\tau} = \mathbf{x}) = \Pr(X_\tau^n = 1 \mid X_0 = \mathbf{x}).$$

Let us use  $\Pr_{\mathbf{x}}(X_\tau^n = 1)$  as a shorthand for  $\Pr(X_\tau^n = 1 \mid X_0 = \mathbf{x})$ . Moreover, let  $\overline{B}$  the event that no bad move occurs in the interval  $[1, \tau]$  and let  $B_t$  denote the event that the first bad move occurs at time  $t \in \{1, \dots, \tau\}$ . Then

$$\Pr_{\mathbf{x}}(X_\tau^n = 1) = \Pr_{\mathbf{x}}(X_\tau^n = 1 \mid \overline{B}) \Pr(\overline{B}) + \sum_{t=1}^{\tau} \Pr_{\mathbf{x}}(X_\tau^n = 1 \mid B_t) \Pr(B_t).$$

Note that  $B_t$  has probability at most  $p$  and  $\overline{B}$  has probability  $(1-p)^\tau$ . Trivially,  $\Pr(X_\tau^n = 1 \mid \overline{B}) = 1$ . Moreover, by Observation 7, given a bad move of player  $i \neq n$  at time  $t_i$ , there is a sequence of time steps  $t_{i+1} < t_{i+2} < \dots < t_n$  such that player  $j \geq i$  is selected at time  $t_j$  and it is not selected further before  $t_{j+1}$ . Therefore, player  $i+1$  plays 0 at time  $t_{i+1}$  with probability  $1-q$  because at that time  $i$  is still playing 0. Similarly, if player  $j$  at time  $t_{j+1}$  is still playing 0, then player  $j+1$  will play 0 with probability  $1-q$ . Hence,

$$\Pr(X_\tau^n \neq 1 \mid B_t) \geq (1-q)^n.$$

Then

$$\begin{aligned}\Pr(X_\tau^n = 1) &\leq (1-p)^\tau + \tau p(1-(1-q)^n) \\ &\leq \frac{1}{1+p\tau} + p\tau \frac{q}{1-q},\end{aligned}$$

where we repeatedly used that  $1-x \leq e^{-x} \leq (1+x)^{-1}$ .

The theorem follows by taking  $p = \frac{1-\varepsilon}{\varepsilon \cdot 2^{n-1}}$  and  $q$  sufficiently small.  $\square$

*Remark.* It is interesting that we can instantiate the abstract game above into the following instance of the BGP game [LSZ08, NSVZ11]:

$$\begin{aligned}u_i(0, s_{-i}) &= 0 \\ u_i(1, s_{-i}) &= \begin{cases} 1, & \text{if } s_1 = \dots = s_{i-1} = 1; \\ -L, & \text{otherwise;} \end{cases}\end{aligned}$$

where  $L$  is a large number. Similarly the update rule described above may be instantiate as a logit update rule with noise  $\beta$ , that corresponds to set

$$p = \frac{1}{1+e^\beta} \quad \text{and} \quad q = \frac{e^{-\beta L}}{e^\beta + e^{-\beta L}}.$$

*Remark.* Note that in the proof of Theorem 6 we considered  $p \approx \frac{1}{R}$  and showed that the corresponding imperfect best-response dynamics does not converge. As a consequence, it may be possible to prove fast convergence to the equilibrium only by taking  $p$  being smaller than  $1/R$ , as done in following section.

### 3.2 Convergence Time

Given the above negative result, we wonder whether there are values of  $p$  for which the convergence of the best-response dynamics is restored. The following theorem states that this occurs when  $p$  is small with respect to parameters  $R, \eta$  and  $e$ .

**Theorem 8.** *For any NBR-solvable game  $G$  and any small  $\delta > 0$  an imperfect best-response dynamics converges to the Nash equilibrium of  $G$  in  $\mathcal{O}(R \cdot e \log e)$  steps with probability at least  $1 - \delta$ , whenever  $p \leq \frac{c}{\eta R \cdot e \log e}$ , for an opportunely chosen constant  $c$ .*

The following two lemmas represent the main tools in the proof of the theorem above. Both lemmas hold for NBR-solvable games as for the more general class of NBR-reducible games. Moreover, in both lemmas we denote with  $X_t$  the random variable that represents the profile of the game after  $t$  steps of an imperfect best-response dynamics. Note also that, for an event  $E$  we denote with  $\Pr_h(E)$  the probability of the event  $E$  conditioned on the initial status being  $h$ .

**Lemma 9.** *For any initial status  $h$ , we have*

$$\Pr_h(X_{t+s} \in G_k \mid X_s \in G_k) \geq 1 - \eta p t, \quad (2)$$

$$\Pr_h(X_{R+s} \in G_{k+1} \mid X_s \in G_k) \geq 1 - \eta p R - \varepsilon. \quad (3)$$

*Proof.* Let the dynamics be in  $G_k$  at time  $s$  and observe that if the dynamics is not in  $G_k$  at time  $t+s$ , then in one of these time steps some selected player played a NBR. Since at every step at most  $\eta$  players are selected, (2) follows from the union bound.

Similarly, if the dynamics is not in  $G_{k+1}$  at time  $t+s$  given that player  $i^{(k)}$  has been selected for update at least once during the interval  $[s+1, s+t]$ , then in one of these time steps some selected player played a NBR. Hence,

$$\Pr_h(X_{t+s} \notin G_{k+1} \mid X_s \in G_k \cap SEL_{i^{(k)},s,t}) \leq \eta t p, \quad (4)$$

where  $SEL_{i,s,t}$  is the event that player  $i$  is selected at least once in the interval  $[s+1, s+t]$ . Now simply observe that

$$\begin{aligned} \Pr_h(X_{t+s} \notin G_{k+1} \mid X_s \in G_k) &\leq \Pr_h(X_{t+s} \notin G_{k+1} \mid X_s \in G_k \cap SEL_{i^{(k)},s,t}) \\ &\quad + (1 - \Pr(SEL_{i^{(k)},s,t})) \\ &\leq \eta t p + (1 - \Pr(SEL_{i^{(k)},s,t})), \end{aligned}$$

where the first inequality follows from the definition of conditional probabilities and the last one uses (4). Since  $\Pr(SEL_{i^{(k)},s,R}) \geq 1 - \varepsilon$  by definition of imperfect best-response dynamics, the lemma follows.  $\square$

**Lemma 10.** *For any initial status  $h$  and  $1 \leq k \leq e$ , we have*

$$\Pr_h(X_{kR} \in G_k) \geq 1 - k \cdot (\eta p R + \varepsilon).$$

*Proof.* Observe that

$$\begin{aligned} \Pr_h(X_{kR} \notin G_k) &\leq \Pr_h(X_{kR} \notin G_k \mid X_{(k-1)R} \in G_{(k-1)R}) \\ &\quad + \Pr_h(X_{(k-1)R} \notin G_{(k-1)R}) \\ &\leq \eta p R + \varepsilon + \Pr_h(X_{(k-1)R} \notin G_{(k-1)R}), \end{aligned}$$

where the first inequality follows from the definition of conditional probabilities and the last one uses (3). Since  $\Pr_h(X_0 \notin G_0) = 0$  the lemma follows by iterating the argument.  $\square$

*Proof of Theorem 8.* Consider an interval  $T$  of length  $R \cdot \left\lceil \frac{\log(2e/\delta)}{\log(1/\varepsilon)} \right\rceil$ . As discussed above, the probability that all players are selected at least once in an interval of length  $T$  is  $\frac{\delta}{2e}$ . The theorem follows by applying Lemma 10 with  $k = e$ ,  $(R, \varepsilon) = (T, \delta/2e)$  and  $p \leq \frac{\delta}{2} \cdot \frac{1}{\eta T e}$ .  $\square$



### 3.3 Incentive Compatibility

Nisan et al. [NSVZ11] showed that eliminating a NBR is *incentive compatible*, i.e., no player benefits by playing at some time step a NBR, for a subclass of NBR-solvable games, namely NBR-solvable game with *clear outcome*. Here, a game is said to have clear outcome if, for every player  $i$ , there is a player-specific elimination sequence such that the following holds: If  $i$  appears the first time in this sequence at position  $k$ , then in the subgame  $G_k$  the profile that maximizes the utility of player  $i = i^{(k)}$  is the Nash equilibrium.

In this section we ask if the incentive compatibility property holds also in presence of “noise”. This means that we are wondering whether the only improvement (if any) can occur by playing a “less imperfect” best-response dynamics, i.e., one whose update rule is  $p'$ -imperfect, with  $p' < p$  (note that every  $p'$ -imperfect update rule with  $p' < p$  is also a  $p$ -imperfect update rule).

**A Negative Answer.** The following theorem shows that the incentive compatibility property is not resistant to the introduction of noise.

**Theorem 11.** *There is a NBR-solvable game with clear outcome and an imperfect best-response dynamics whose update rule is not incentive compatible.*

*Proof.* Consider the following game  $G$  with clear outcome (the “gray profile”)

	left	right
top	$c + 2, 1$	$1, 0$
bottom	$0, 0$	$0, c$

and suppose to run the logit dynamics for  $G$  (we already noted that the logit dynamics is an example of imperfect best-response dynamics). We will show that the column player has a better expected payoff by playing always strategy “right”.

The above game is a *potential game* and the potential  $\Phi$  is

	left	right
top	$c + 2$	$c + 1$
bottom	$0$	$c$

It is known (see, for example, [Blu93, AFPP10]) that in this case the logit dynamics converges to a distribution on the set of profiles such that the probability of a profile  $\mathbf{x}$  is proportional to  $e^{\beta\Phi(\mathbf{x})}$ . Hence, the expected utility of the column player when she plays according to the logit update rule is

$$\frac{1 \cdot e^{\beta(c+2)} + c \cdot e^{\beta c}}{1 + e^{\beta c} + e^{\beta(c+1)} + e^{\beta(c+2)}} < \frac{e^{2\beta} + c}{1 + e^{\beta} + e^{2\beta}}. \quad (5)$$

If instead the column player always plays strategy “right”, then her expected payoff is determined by the logit dynamics on the corresponding subgame and it is equal to

$$\frac{c \cdot e^{\beta c}}{e^{\beta c} + e^{\beta(c+1)}} = \frac{c}{1 + e^{\beta}}. \quad (6)$$

Since the right-hand side of (5) is smaller than (6) for  $c \geq 1 + e^\beta$ , the lemma follows.  $\square$

**Sufficient Conditions.** As done for convergence, we now investigate for sufficient conditions for incentive compatibility. We will assume that utilities are non-negative: note that there are a lot of update rules that are invariant with respect to the actual value of the utility function and thus, in these cases, this assumption is without loss of generality. Moreover when we say that player  $i = i^{(k)}$  we are assuming that  $G_k$  is the first subgame in which  $i$  is asked to eliminate a NBR strategy in her elimination sequence.

It turns out that we need a “quantitative” version of the clear outcome property, i.e., that whenever the player  $i$  has to eliminate a NBR her utility in the Nash equilibrium is sufficiently larger than the utility of any other profile in the subgame she is actually playing. Specifically, we have the following theorem.

**Theorem 12.** *For any NBR-solvable game  $G$  and any small  $\delta > 0$ , playing according to a  $p$ -imperfect rule is incentive compatible for player  $i = i^{(k)}$  as long as  $p \leq \frac{c}{\eta R \cdot e \log e}$ , for an opportune chosen constant  $c$ , the dynamics run for  $\Omega(R \cdot e \log e)$  and*

$$u_i(NE) \geq \frac{1}{1 - 2\delta} \left( 2\delta \cdot u_i^* + u_i^k \right),$$

where  $u_i(NE)$  is the utility of  $i$  in the Nash equilibrium,  $u_i^k = \max_{\mathbf{x} \in G^{(k)}} u_i(\mathbf{x})$  and  $u_i^* = \max_{\mathbf{x} \in G} u_i(\mathbf{x})$ .

We can summarize the intuition behind the proof of Theorem 12 as follows:

- If player  $i$  always update according the  $p$ -imperfect update rule, then the game will be in the Nash equilibrium for a lot of time steps and hence her expected utility almost coincides with the Nash equilibrium utility;
- Suppose, instead, player  $i$  does not update according a  $p$ -imperfect update rule. Notice that elimination of strategies up to  $G_k$  is not affected by what player  $i$  does. Therefore profiles of  $G \setminus G_k$  will be played only for a small number of times (but  $i$  can gain the highest possible utility from these profiles), whereas for the rest of the time the game will be in a profile of  $G_k$ .

Let us now formalize this idea. We start with the following lemma.

**Lemma 13.** *For any initial status  $h$ , any  $1 \leq k \leq e$  and any  $t \geq kR$ , we have*

$$\Pr_h(X_t \in G_k) \geq 1 - \eta p \cdot (t - \ell k R) - k \cdot (\eta p R + \varepsilon),$$

where  $\ell$  is the largest integer such that  $t \geq \ell k R$ .

*Proof.* We have

$$\Pr_h(X_t \notin G_k) \leq \Pr_h(X_t \notin G_k \mid X_{\ell k R} \in G_k) + \Pr_h(X_{\ell k R} \notin G_k).$$

From Lemma 9 we have

$$\Pr_h(X_t \notin G_k \mid X_{\ell k R} \in G_k) \leq \eta p \cdot (t - \ell k R).$$

Moreover, let  $h'$  be the status that contains every information collected in the first  $(\ell - 1)kR$  steps of the dynamics. Then by Lemma 10 we have

$$\Pr_h(X_{\ell k R} \notin G_k) = \Pr_{h'}(X_{k R} \notin G_k) \leq k \cdot (\eta p R + \varepsilon). \quad \square$$

*Remark.* Observe that that Lemma 13 holds even if only players  $i^{(1)}, \dots, i^{(k)}$  are updating according a  $p$ -imperfect update rule.

*Proof of Theorem 12.* Let us start by computing the expected utility of  $i$ , given that all players are playing according to the  $p$ -imperfect update rule. Let  $T$  and  $p$  as in the proof of Theorem 8. Then, by applying Lemma 13 with  $k = e$  and  $(R, \varepsilon) = (T, \delta/2e)$  we have for any  $t = \Omega(R \cdot e \log e)$

$$\Pr_h(X_t \in \hat{G}) \geq 1 - 2\delta.$$

Hence, the expected utility of  $i$  will be at least  $(1 - 2\delta) \cdot u_i(NE)$ .

Suppose now that  $i$  does not play a  $p$ -imperfect update rule. Similarly as done above, we let  $T = R \cdot \left\lceil \frac{\log(2k/\delta)}{\log(1/\varepsilon)} \right\rceil$  and then, by applying Lemma 13 with  $(R, \varepsilon) = (T, \delta/2k)$  we obtain

$$\Pr_h(X_t \notin G_k) \leq 2\delta.$$

Hence, the expected utility of  $i$  will be at most  $2\delta u_i^* + u_i^k$  and the theorem follows.  $\square$

## 4 NBR-reducible games

In the previous section we focused on NBR-solvable games and pure Nash equilibria. Now, we will see that some of the ideas developed there can be extended in order to handle NBR-reducible games and more generic equilibrium concepts. In particular, we will see that for a wide class of equilibrium concepts, the convergence of an imperfect best-response dynamics for a NBR-reducible game  $G$  can be analyzed by considering a restriction of this dynamics to the reduced game  $\hat{G}$ .

**The Dynamics as a Markov Chain.** Let us start by introducing some useful notation. We say that the game is in a *pair status-profile*  $(h, \mathbf{x})$  if  $s$  the set of informations available and  $\mathbf{x}$  is the profile currently played. We denote with  $H$  the set of all pairs status-profile  $(h, \mathbf{x})$  and with  $\hat{H}$  only the ones with  $\mathbf{x} \in \hat{G}$ . Let  $X_t$  be the random variable that represents the pair status-profile

$(h, \mathbf{x})$  in which the game is after  $t$  steps of the original dynamics. Then, for every  $(h, \mathbf{x}), (z, \mathbf{y}) \in H$  we set

$$P((h, \mathbf{x}), (z, \mathbf{y})) = \Pr(X_1 = (z, \mathbf{y}) \mid X_0 = (h, \mathbf{x})).$$

That is,  $P$  is the transition matrix of a Markov chain on state space  $H$  and it describes exactly the evolution of the dynamics. Note that we are not restricting the dynamics to be memoryless, since in the status we can save the history of all previous iterations. For a set  $A \subseteq H$  we also denote  $P((h, \mathbf{x}), A) = \sum_{(z, \mathbf{y}) \in A} P((h, \mathbf{x}), (z, \mathbf{y}))$ .

**The Restricted Dynamics.** As told before, we will compare the original dynamics with a specific restriction on the subset  $\hat{H}$  of pairs status–profile. Now we describe how this restriction is obtained. Henceforth, when we will refer to the restricted dynamics, we will use  $\hat{X}_t$  and  $\hat{P}$  in place of  $X_t$  and  $P$ . Then, the restricted dynamics is described by a Markov chain on state space  $H$  with transition matrix  $\hat{P}$  such that for every  $(h, \mathbf{x}), (z, \mathbf{y}) \in H$

$$\hat{P}((h, \mathbf{x}), (z, \mathbf{y})) = \begin{cases} \frac{P((h, \mathbf{x}), (z, \mathbf{y}))}{P((h, \mathbf{x}), \hat{H})}, & \text{if } (h, \mathbf{x}), (z, \mathbf{y}) \in \hat{H}; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the restricted dynamics is exactly the same as the original one except that the first never leaves the subgame  $\hat{G}$ , whereas in the latter, at each time step, there is probability at most  $p$  to leave this subgame. The following lemma quantifies this similarity, by showing that, for every  $(h, \mathbf{x}) \in \hat{H}$ , the *total variation distance* (TV)<sup>2</sup> between the original and the restricted dynamics starting from  $(h, \mathbf{x})$  is small.

**Lemma 14.** *For every  $(h, \mathbf{x}) \in \hat{H}$ ,*

$$\left\| P^t((h, \mathbf{x}), \cdot) - \hat{P}^t((h, \mathbf{x}), \cdot) \right\| \leq \eta p t. \quad (7)$$

*Proof.* The proof is by induction on  $t$ . The base case is  $t = 1$  for which the set of pairs status–profile  $(z, \mathbf{y})$  such that  $P((h, \mathbf{x}), (z, \mathbf{y})) > \hat{P}((h, \mathbf{x}), (z, \mathbf{y}))$  is exactly  $\bar{H} = H \setminus \hat{H}$  and hence

$$\begin{aligned} \left\| P((h, \mathbf{x}), \cdot) - \hat{P}((h, \mathbf{x}), \cdot) \right\| &= \sum_{(z, \mathbf{y}) \in \bar{H}} \left( P((h, \mathbf{x}), (z, \mathbf{y})) - \hat{P}((h, \mathbf{x}), (z, \mathbf{y})) \right) \\ &= P((h, \mathbf{x}), \bar{H}) \leq \eta p, \end{aligned}$$

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<sup>2</sup>See Appendix B for a review of the main properties of the total variation distance.

where the last inequality follows from Lemma 9. Furthermore

$$\begin{aligned}
& \left\| P^t((h, \mathbf{x}), \cdot) - \hat{P}^t((h, \mathbf{x}), \cdot) \right\| \leq \\
& \quad (\text{TV triangle inequality}) \leq \left\| P((h, \mathbf{x}), \cdot) P^{t-1} - \hat{P}((h, \mathbf{x}), \cdot) P^{t-1} \right\| \\
& \quad \quad + \left\| \hat{P}((h, \mathbf{x}), \cdot) P^{t-1} - \hat{P}((h, \mathbf{x}), \cdot) \hat{P}^{t-1} \right\| \\
& \quad (\text{TV monotonicity}) \leq \left\| P((h, \mathbf{x}), \cdot) - \hat{P}((h, \mathbf{x}), \cdot) \right\| \\
& \quad \quad + \sup_{(z, \mathbf{y}) \in \hat{H}} \left\| P^{t-1}((z, \mathbf{y}), \cdot) - \hat{P}^{t-1}((z, \mathbf{y}), \cdot) \right\| \\
& \quad (\text{induction and Lemma 9}) \leq \eta p + \eta p(t-1) = \eta p t. \quad \square
\end{aligned}$$

**Equilibria and Convergence.** Several and different equilibrium concepts of independent interest has been introduced, as, for example, sink equilibria [GMV05], correlated equilibria [Aum74] and logit equilibria [AFPP10]. We would like to give a generic result that holds for each one of these equilibria and for any other equilibrium concept at which one may be interested. For this reason in the following we will consider a *generic equilibrium*, that is either a set of pairs status–profile or a distribution on these pairs. Note that each one of the equilibria above described are included in this definition. However, the definition includes also equilibrium concepts like “the first profile that is visited for 10 times” or “the first cycle of length 4 visited”. We remark that in this case it is critical that the equilibrium is defined on the pairs status–profile and not just on the profiles: indeed, the status can remember the history of the game and identify such equilibria, whereas they are impossible to recognize if we only know the current profile.

At the twofold definition of a generic equilibrium corresponds a twofold meaning of *convergence time*. Indeed, if the equilibrium is represented by a set of pairs status–profile, then we are interested in the first time step in which the game has reached this set. In the case that the equilibrium is given by a distribution, then we are interested in the first time step in which this equilibrium distribution is close to the the distribution on the set of profiles generated by the dynamics.

**The Main Theorem.** Let us denote with  $\tau$  the time the restricted dynamics takes to converge to a generic equilibrium  $E$ . Then we have the following theorem.

**Theorem 15.** *For an NBR-reducible game and any small  $\delta > 0$  an imperfect best-response dynamics converges to  $E$  in  $\mathcal{O}(R \cdot e \log e + \tau)$  steps with probability at least  $1 - \delta$ , whenever  $p \leq \min \left\{ \frac{c_1}{\eta R \cdot e \log e}, \frac{c_2}{\eta \tau} \right\}$ , for opportunely chosen constants  $c_1, c_2$ .*

*Proof.* We will show that the dynamics will be in  $\hat{H}$  after  $\mathcal{O}(R \cdot e \log e)$  with probability at least  $1 - \delta/2$ ; moreover, if the dynamics is in  $\hat{H}$  after a number  $t$  of

steps, then it will converge to the equilibrium in further  $\tau$  steps with probability at least  $1 - \delta/2$ . Hence, the probability that the dynamics does not converges in  $\mathcal{O}(R \cdot e \log e + \tau)$  steps will be at most  $\delta$  and the theorem follows.

Specifically, consider an interval  $T$  of length  $R \cdot \left\lceil \frac{\log(4e/\delta)}{\log(1/\varepsilon)} \right\rceil$ . By applying Lemma 10 with  $k = e$ ,  $(R, \varepsilon) = (T, \delta/4e)$  and  $p \leq \frac{\delta}{4} \cdot \frac{1}{\eta T e}$  we have that for every  $(h, \mathbf{x}) \in H$

$$\Pr(X_{eT} \in \hat{H} \mid X_0 = (h, \mathbf{x})) \geq 1 - \delta/2.$$

Finally, note that the probability that, for every  $t > 0$ , the dynamics converges to the equilibrium in  $t + \tau$  steps given that after  $t$  steps it is in  $(z, \mathbf{y}) \in \hat{H}$ , is the same as if we assume the dynamics starts in  $(z, \mathbf{y})$ , i.e., it is equivalent to the probability that the dynamics converges to the equilibrium in  $\tau$  steps from  $(z, \mathbf{y})$ . If the equilibrium concept at which the restricted dynamics converges after  $\tau$  steps is a distribution  $\pi$  on the pairs status-profile, then, from (7), the distribution after  $\tau$  steps of the original dynamics is  $\pi$  except for an amount of probability of at most  $\eta p \tau$ . On the other side, if the equilibrium concept at which the restricted dynamics converges after  $\tau$  steps is a set  $A$  of pairs status-profile, then, from (7) we have

$$\Pr(X_\tau \in A) \geq \Pr(\hat{X}_\tau \in A) - \mu p \tau = 1 - \mu p \tau,$$

and hence, after  $\tau$  steps, the original dynamics is in  $A$  except with probability at most  $\mu p \tau$ . Then, by Lemma 14 and by taking  $p \leq \frac{\delta}{2} \cdot \frac{1}{\eta \tau}$ , the probability that the original dynamics converges to the equilibrium in  $\tau$  steps starting from  $(z, \mathbf{y})$  is at least  $1 - \delta/2$ .  $\square$

**Examples.** Here we give several examples in which we adopt Theorem 15 to bound the rate of convergence of an imperfect best-response dynamics to different kind of equilibria. Specifically, consider a NBR-reducible game  $G$ , as for example the one described in (1), and consider an imperfect best-response dynamics, as for example the logit dynamics. Suppose we are interested in evaluating the time the dynamics takes to reach a sink equilibrium: note that, since all profiles not in  $\hat{G}$  contain iteratively dominated strategies, the sink equilibria of  $G$  are exactly the sink equilibria of  $\hat{G}$  and then, by Theorem 15, it is sufficient to analyze what happens in this subgame.

Suppose instead that we are interested in the time that the dynamics takes before the distribution over the profile generated by the dynamics is close to the one generated by a correlated equilibrium. As before profiles not in  $\hat{G}$ , since they contain iteratively dominated strategies, do not appear in the support of any correlated equilibria. Then, again, by Theorem 15, we can simply analyze what happen in  $\hat{G}$ .

For another interesting example, suppose we are wondering about the convergence to the logit equilibrium. Note that, differently from what happen for correlated equilibria, the logit equilibrium assigns non-zero probability to profiles not in  $\hat{G}$ . However, it is not difficult to show (see Appendix C) that the

logit equilibrium of  $G$  is very close to the logit equilibrium of  $\hat{G}$ . Hence, even in this case, by Theorem 15 bounds on the convergence can be easily given by focusing on  $\hat{G}$ .

Finally, if we consider equilibria like “the first profile that is visited for 10 times”, then the time that the dynamics takes to converge to these equilibria is obviously less than if we restrict the profile to being in  $\hat{G}$  and, hence, by Theorem 15, it is sufficient to analyze the restricted dynamics.

## References

- [AFN10] Carlos Alós-Ferrer and Nick Netzer. The logit-response dynamics. *Games and Economic Behavior*, 68(2):413–427, 2010.
- [AFPP10] Vincenzo Auletta, Diodato Ferraioli, Francesco Pasquale, and Giuseppe Persiano. Mixing time and stationary expected social welfare of logit dynamics. In *3rd Int. Symp. on Algorithmic Game Theory (SAGT’10)*, volume 6386 of *LNCS*, pages 54–65. Springer, 2010.
- [Aum74] Robert J. Aumann. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1(1):67–96, 1974.
- [Blu93] Lawrence E. Blume. The statistical mechanics of strategic interaction. *Games and Economic Behavior*, 5:387–424, 1993.
- [GMV05] Michel Goemans, Vahab Mirrokni, and Adrian Vetta. Sink equilibria and convergence. In *Proc. of the 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS’05)*, pages 142–154. IEEE, 2005.
- [KMR93] Michihiro Kandori, George J. Mailath, and Rafael Rob. Learning, mutation, and long run equilibria in games. *Econometrica*, 61(1):29–56, 1993.
- [KR95] Michihiro Kandori and Rafael Rob. Evolution of equilibria in the long run: A general theory and applications. *Journal of Economic Theory*, 65(2):383 – 414, 1995.
- [LSZ08] Hagay Levin, Michael Schapira, and Aviv Zohar. Interdomain routing and games. In *Proceedings of the 40th annual ACM symposium on Theory of computing, STOC ’08*, pages 57–66, New York, NY, USA, 2008. ACM.
- [NSVZ11] Noam Nisan, Michael Schapira, Gregory Valiant, and Aviv Zohar. Best-response mechanisms. In *Innovations in Computer Science (ICS)*, pages 155–165, 2011.
- [You93] H. Peyton Young. The evolution of conventions. *Econometrica*, 61(1):57–84, 1993.

## A Logit Dynamics

The *logit dynamics* for a game  $G$  runs as follows: at every time step (i) Select one player  $i \in [n]$  uniformly at random; (ii) Update the strategy of player  $i$  according to the *Boltzmann distribution* with parameter  $\beta$  over the set  $S_i$  of her strategies. That is, a strategy  $s_i \in S_i$  will be selected with probability

$$\sigma_i(s_i \mid \mathbf{x}_{-i}) = \frac{1}{Z_i(\mathbf{x}_{-i})} e^{\beta u_i(\mathbf{x}_{-i}, s_i)}, \quad (8)$$

where  $u_i$  is the utility function of the player  $i$ ,  $\mathbf{x}_{-i} \in \{0, 1\}^{n-1}$  is the profile of strategies played at the current time step by players different from  $i$ ,  $Z_i(\mathbf{x}_{-i}) = \sum_{z_i \in S_i} e^{\beta u_i(\mathbf{x}_{-i}, z_i)}$  is the normalizing factor, and  $\beta \geq 0$ . From (8), it is easy to see that for  $\beta = 0$  player  $i$  selects her strategy uniformly at random, for  $\beta > 0$  the probability is biased toward strategies promising higher payoffs, and for  $\beta$  that goes to  $\infty$  player  $i$  chooses her best response strategy (if more than one best response is available, she chooses one of them uniformly at random).

The above dynamics defines a *Markov chain*  $\{X_t\}_{t \geq 0}$  with the set of strategy profiles as state space, and where the probability  $P(\mathbf{x}, \mathbf{y})$  of a transition from profile  $\mathbf{x} = (x_1, \dots, x_n)$  to profile  $\mathbf{y} = (y_1, \dots, y_n)$  is zero if  $H(\mathbf{x}, \mathbf{y}) \geq 2$  and it is  $\frac{1}{n} \sigma_i(y_i \mid \mathbf{x}_{-i})$  if the two profiles differ exactly at player  $i$ . More formally, we can define the logit dynamics as follows.

**Definition 16** (Logit dynamics [Blu93]). *Let  $G$  be a game and let  $\beta \geq 0$ . The logit dynamics for  $G$  is the Markov chain  $\mathcal{M}_\beta = (\{X_t\}_{t \geq 0}, S, P)$  where  $S$  is the set of profiles of  $G$  and*

$$P(\mathbf{x}, \mathbf{y}) = \frac{1}{n} \cdot \begin{cases} \sigma_i(y_i \mid \mathbf{x}_{-i}), & \text{if } \mathbf{y}_{-i} = \mathbf{x}_{-i} \text{ and } y_i \neq x_i; \\ \sum_{i=1}^n \sigma_i(y_i \mid \mathbf{x}_{-i}), & \text{if } \mathbf{y} = \mathbf{x}; \\ 0, & \text{otherwise;} \end{cases} \quad (9)$$

where  $\sigma_i(y_i \mid \mathbf{x}_{-i})$  is defined in (8).

The Markov chain defined by (9) is ergodic. Hence, from every initial profile  $\mathbf{x}$  the distribution  $P^t(\mathbf{x}, \cdot)$  over profiles after the chain has taken  $t$  steps starting from  $\mathbf{x}$  will eventually converge to a *stationary distribution*  $\pi$  as  $t$  tends to infinity. As in [AFPP10], we call the stationary distribution  $\pi$  of the Markov chain defined by the logit dynamics on a game  $G$ , the *logit equilibrium* of  $G$ .

For the class of potential games the stationary distribution is the well-known *Gibbs measure*.

**Theorem 17** ([Blu93]). *If  $G$  is a potential game with potential function  $\Phi$ , then the stationary distribution  $\pi$  of the Markov chain given by (9) is*

$$\pi(\mathbf{x}) = \frac{1}{Z} e^{-\beta \Phi(\mathbf{x})}$$

, where  $Z = \sum_{\mathbf{y} \in S} e^{-\beta \Phi(\mathbf{y})}$  is the normalizing constant.



## B Total Variation Distance

The *total variation distance* between distributions  $\mu$  and  $\hat{\mu}$  on an enumerable state space  $\Omega$  is

$$\|\mu - \hat{\mu}\| := \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \hat{\mu}(x)| = \sum_{\substack{x \in \Omega \\ \mu(x) > \hat{\mu}(x)}} \mu(x) - \hat{\mu}(x).$$

Note that the total variation distance satisfies the usual triangle inequality of distance measures, i.e.,

$$\|\mu - \hat{\mu}\| \leq \|\mu - \mu'\| + \|\mu' - \hat{\mu}\|.$$

for every distribution  $\mu'$ . Moreover, the following monotonicity properties hold:

$$\|\mu P - \hat{\mu} P\| \leq \|\mu - \hat{\mu}\|, \quad (10)$$

$$\|\mu P - \mu \hat{P}\| \leq \sup_{x \in \Omega} \|P(x, \cdot) - \hat{P}(x, \cdot)\|, \quad (11)$$

$$\|\mu P - \hat{\mu} P\| \leq \sup_{x, y \in \Omega} \|P(x, \cdot) - P(y, \cdot)\|, \quad (12)$$

where  $P$  and  $\hat{P}$  are stochastic matrices. Indeed, as for (10) we have

$$\begin{aligned} \|\mu P - \hat{\mu} P\| = \|(\mu - \hat{\mu})P\| &= \frac{1}{2} \sum_{x \in \Omega} \left| (\mu(x) - \hat{\mu}(x)) \sum_{y \in \Omega} P(x, y) \right| \\ &\leq \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \hat{\mu}(x)| \sum_{y \in \Omega} P(x, y) \\ &= \|\mu - \hat{\mu}\|. \end{aligned}$$

As for (11) we observing that

$$\begin{aligned} \|\mu P - \mu \hat{P}\| = \|\mu(P - \hat{P})\| &= \frac{1}{2} \sum_{x \in \Omega} \left| \mu(x) \sum_{y \in \Omega} (P(x, y) - \hat{P}(x, y)) \right| \\ &\leq \sum_{x \in \Omega} \mu(x) \left( \frac{1}{2} \sum_{y \in \Omega} |P(x, y) - \hat{P}(x, y)| \right) \\ &\leq \sup_{x \in \Omega} \|P(x, \cdot) - \hat{P}(x, \cdot)\|. \end{aligned}$$

Finally, for (12) we have

$$\begin{aligned} \|\mu P - \hat{\mu} P\| &= \left\| \sum_{z \in \Omega} \mu(z) \sum_{w \in \Omega} \hat{\mu}(w) (P(z, \cdot) - P(w, \cdot)) \right\| \\ &\leq \sum_{z \in \Omega} \mu(z) \sum_{w \in \Omega} \hat{\mu}(w) \|P(z, \cdot) - P(w, \cdot)\| \\ &\leq \sup_{x, y \in \Omega} \|P(x, \cdot) - P(y, \cdot)\|. \end{aligned}$$

## C Logit Equilibria and NBR-Reducible Games

Let  $G$  be a game NBR-reducible to  $\hat{G}$ . Let  $\pi$  be the stationary distributions of the logit dynamics for  $G$  and  $\hat{\pi}$  be the stationary distribution of the restriction of this dynamics to  $\hat{G}$ . Then the following lemma holds for  $\beta$  large enough.

**Lemma 18.** *For every  $\delta > 0$ ,*

$$\|\pi - \hat{\pi}\| \leq \delta,$$

*for  $\beta$  sufficiently large.*

*Proof.* Let  $\tau = \hat{t}_{\text{mix}}(\delta/8)$  be the mixing time of the restricted chain. Consider first two copies of the chain starting in profiles  $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in \hat{G}$  and bound the total variation after  $\tau$  time steps:

$$\begin{aligned} \|P^\tau(\hat{\mathbf{x}}, \cdot) - P^\tau(\hat{\mathbf{y}}, \cdot)\| &\leq \|P^\tau(\hat{\mathbf{x}}, \cdot) - \hat{P}^\tau(\hat{\mathbf{x}}, \cdot)\| + \|\hat{P}^\tau(\hat{\mathbf{x}}, \cdot) - \hat{\pi}\| \\ &\quad + \|\hat{\pi} - \hat{P}^\tau(\hat{\mathbf{y}}, \cdot)\| + \|\hat{P}^\tau(\hat{\mathbf{y}}, \cdot) - P^\tau(\hat{\mathbf{y}}, \cdot)\| \\ &\leq 4 \cdot \frac{\delta}{8} = \delta/2, \end{aligned}$$

where the last inequality is due to Lemma 14 by taking  $\beta$  sufficiently large. Consider an interval  $T$  of length  $R \cdot \left\lceil \frac{\log(8e/\delta)}{\log(1/\varepsilon)} \right\rceil$ . By applying Lemma 10 with  $k = e$ ,  $(R, \varepsilon) = (T, \delta/4e)$  and  $\beta$  sufficiently large we have that for every  $\mathbf{x} \in G$

$$\Pr_{\mathbf{x}}(X_{eT} \notin \hat{G}) \leq \delta/8.$$

Let  $t^* = eT + \tau$  and  $Q = P^{eT}$ . Then, for every  $\mathbf{x}, \mathbf{y} \in G$

$$\begin{aligned} \|\pi - P^{t^*}(\mathbf{y}, \cdot)\| &\leq \|P^{t^*}(\mathbf{x}, \cdot) - P^{t^*}(\mathbf{y}, \cdot)\| = \|Q(\mathbf{x}, \cdot)P^\tau - Q(\mathbf{y}, \cdot)P^\tau\| \\ \text{(triangle inequality)} &\leq \|Q(\mathbf{x}, \cdot)P^\tau - \hat{Q}(\mathbf{x}, \cdot)P^\tau\| + \|\hat{Q}(\mathbf{x}, \cdot)P^\tau - \hat{Q}(\mathbf{y}, \cdot)P^\tau\| \\ &\quad + \|\hat{Q}(\mathbf{y}, \cdot)P^\tau - Q(\mathbf{y}, \cdot)P^\tau\|, \end{aligned}$$

where, for every  $\mathbf{x}, \mathbf{y} \in G$ , we set

$$\hat{Q}(\mathbf{x}, \mathbf{y}) = \begin{cases} \frac{Q(\mathbf{x}, \mathbf{y})}{Q(\mathbf{x}, \hat{H})}, & \text{if } \mathbf{x}, \mathbf{y} \in \hat{G}; \\ 0, & \text{otherwise.} \end{cases}$$

By (10) we obtain

$$\|Q(\mathbf{x}, \cdot)P^\tau - \hat{Q}(\mathbf{x}, \cdot)P^\tau\| \leq \|Q(\mathbf{x}, \cdot) - \hat{Q}(\mathbf{x}, \cdot)\| \leq \Pr_{\mathbf{x}}(X_{eT} \notin \hat{G}) \leq \delta/8.$$

By (12) we obtain

$$\|\hat{Q}(\mathbf{x}, \cdot)P^\tau - \hat{Q}(\mathbf{y}, \cdot)P^\tau\| \leq \max_{\hat{\mathbf{x}}, \hat{\mathbf{y}} \in \hat{G}} \|P^\tau(\hat{\mathbf{x}}, \cdot) - P^\tau(\hat{\mathbf{y}}, \cdot)\| \leq \delta/2.$$

and hence  $\|\pi - P^{t^*}(\mathbf{y}, \cdot)\| \leq 3\delta/4$ . Finally, for every  $\hat{\mathbf{x}} \in \hat{G}$ , by triangle inequality

$$\begin{aligned} \|\pi - \hat{\pi}\| &\leq \left\| \pi - P^{t^*}(\hat{\mathbf{x}}, \cdot) \right\| + \left\| P^{t^*}(\hat{\mathbf{x}}, \cdot) - \hat{P}^{t^*}(\hat{\mathbf{x}}, \cdot) \right\| + \left\| \hat{P}^{t^*}(\hat{\mathbf{x}}, \cdot) - \hat{\pi} \right\| \\ &\leq 3\delta/4 + \delta/8 + \delta/8 = \delta. \end{aligned}$$

□